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MEASURABLE SELECTIONS OF EXTREMA

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Let $f: X \times Y \to R$. We prove two theorems concerning the existence of a measurable function φ such that $f(x, \varphi(x)) = \inf_y f(x, y)$. The first concerns Borel measurability and the second concerns absolute (or universal) measurability. These results are related to the existence of measurable projections of sets $S \subset X \times Y$.

Among other applications these theorems can be applied to the problem of finding measurable Bayes procedures according to the usual procedure of minimizing the a posteriori risk. This application is described here and a counterexample is given in which a Borel measurable Bayes procedure fails to exist.

1. Introduction. Let f be a bounded real-valued Borel measurable function defined on the unit square. Suppose that for each $x \in [0, 1]$, there is at least one y for which $f(x, y) = \inf_z f(x, z)$. Then it has long been known (see Section 5) that there may be no Borel measurable way of choosing one such y for each x; that is, no Borel measurable φ on $[0, 1]$ for which $f(x, \varphi(x)) = \inf_z f(x, z)$, $x \in [0, 1]$. The question of the existence of such measurable φ arises naturally in Statistical Decision Theory, both in the case of Bayes procedures and maximum likelihood procedures.

In a more general setting than the unit square, Corollary 1 (Section 2) gives sufficient conditions for the existence of a Borel measurable φ. Theorem 2 states that an absolutely measurable φ always exists. In either case, the method of proof amounts to applying a known selection theorem. The first of the selection theorems is difficult, but the work here is a straightforward utilization of the methods of Polish Topology.

The relation of these results to the existence of measurable Bayes procedures is discussed in Section 4. In Section 5, an example is described for which a Borel measurable Bayes procedure does not exist.

The methods of this paper can also be used to prove measurability of other types of statistical procedures. One of the authors has used them to prove measurability of the best invariant estimator in a certain situation (see Brown (1966) Lemma 2.2.1).

2. Borel measurability. The following conventions will be observed throughout all sections of the paper. If M is a metric space, $\mathscr{B}(M)$ the Borel σ-field is the smallest sigma-field containing the closed subsets of M. A *Borel measurable function* f is a function whose domain is a metric space, which takes its values in some metric space M, and which is Borel measurable in the usual sense: $B \in \mathscr{B}(M)$ implies $f^{-1}(B) \in \mathscr{B}(\text{domain } f)$.

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If 0 is a set of ordered pairs, the projection of 0, or proj (0), is the set of all first coordinates of members of 0.

If $E \subseteqq U \times V$, where U, V are metric spaces, S will be said to be a *Borel selection* of E("ein uniformisierung") provided

(i) S is a Borel set;
(ii) $S \subseteqq E$;
(iii) For each $u \in U$, the section $S_u = \{v \in V \mid (u, v) \in S\}$ contains at most one point;
(iv) proj $(S) =$ proj (E).

Corresponding to each selection S is the function ρ_s, which assigns to each $u \in$ proj (E) the second coordinate of the unique member of S with first coordinate u. Thus, $(u, \rho_s(u)) \in E$, for all $u \in$ proj (E).

The first purpose of this section is to state and prove a selection theorem which is a slight generalization of a theorem due to E. A. Stschegolkow (Arsenin and Ljapunov (1955) Theorem 39). Stschegolkow, utilizing the separation principles of P. S. Novikoff, showed that if a Borel set E in the unit square has the property that each of its vertical sections can be expressed as a countable union of closed sets, then E has a Borel selection.

THEOREM 1. *Let U, V be complete separable metric spaces and $E \subseteqq U \times V$ be a Borel set. If, for each $u \in U$, the section E_u is σ-compact there is a Borel selection, S, of E. Further* proj (E) *is a Borel set and ρ_s is a Borel measurable function defined on* proj (E).

PROOF. The argument, which takes place in three stages, shows how the result of Stschegolkov (Arsenin and Ljapunov, 1955, Theorem 39) can be used to obtain Borel selections for the more general spaces of the theorem.

1. First, let $U = [0, 1]$ and V be the Hilbert cube,

$$H = [0, 1] \times [0, 1] \times \cdots .$$

There is a φ continuous from $[0, 1]$ *onto* H. (See Kuratowski (1958) page 384, for example.) Let T be the Borel measurable function defined by

$$T\colon (u, y) \to (u, \varphi(y)) \quad (u, y) \in [0, 1] \times [0, 1] .$$

Then $D = T^{-1}(E)$ is a Borel subsets of the unit square with $D_u = \varphi^{-1}(E_u)$ always a countable union of closed sets by the continuity of φ. The theorem of Stschegolkow applies to give a Borel selection R for D. Then $S = T(R)$ is a Borel selection for E. Indeed, it is straightforward to check that S satisfies (ii), (iii), (iv), and that T is 1–1 restricted to R. In complete separable metric spaces the 1–1 Borel measurable image of a Borel set is a Borel set (Kuratowski (1958) page 397), so that S is a Borel set. This proves the first statement of the theorem for $U = [0, 1]$ and $V = H$.

2. Let $U = [0, 1]$ and V be an arbitrary complete separable metric space. By

the well-known metrization theorem of Urysohn, V is homeomorphic to a subspace of H. If λ is one such homeomorphism, let

$$T: (u, v) \to (u, \lambda(v)) \quad (u, v) \in [0, 1] \times V .$$

Then $F = T(E)$ is a Borel subset of $[0, 1] \times H$ (Borel measurable, 1–1 image of a Borel set) and for each $u \in [0, 1]$, F_u is a countable union of sets which are compact in $\lambda(V)$. Sets compact in $\lambda(V)$ are also compact in H and, from step one, there is a Borel selection Q for F. Then $T^{-1}(Q)$ is a Borel selection for E.

3. Let U be an arbitrary complete separable metric space. Then U can be represented as the range of a 1–1 continuous function f defined on a closed subset M of the subspace of the irrational numbers in $[0, 1]$ (Kuratowski (1958) page 354). Define T by

$$T: (x, v) \to (f(x), v) , \quad (x, v) \in M \times V$$

and set $D = T^{-1}(E)$. By embedding $M \times V$ in $[0, 1] \times V$ and applying step 2, it follows that D has a Borel selection R. Finally, since T is 1–1 Borel measurable, $S = T(R)$ is a Borel set. It is then easily checked that S is a Borel selection of E.

This completes the proof of the first statement of the theorem. To show the second statement, let E satisfy the hypothesis of the theorem and S be a Borel selection of E. Each section S_u of S contains at most one point. It follows that for any Borel set $B \subseteqq V$, the set

$$\{u \in \operatorname{proj}(E) \mid \rho_s(u) \in B\} = \operatorname{proj}(S \cap (U \times B))$$

is a 1–1 Borel measurable image of the Borel set $S \cap (U \times B)$ and hence is Borel. This implies that proj (E) is a Borel set and that ρ_s is Borel measurable. This completes the proof of Theorem 1.

In the statement of the following corollary, where D is the domain of a real-valued function f of two variables, the infimum of the set of reals $\{f(x, y) \mid y \in D_x\}$ is abbreviated $\inf f_x$. The infimum of a set of numbers which is not bounded below will be taken to be $-\infty$.

COROLLARY 1. *Let X, Y be complete separable metric spaces and f be a real-valued Borel measurable function defined on a Borel subset D of $X \times Y$.*

Suppose that for each $x \in$ proj (D), the section D_x is σ-compact and $f(x, \cdot)$ is lower semi-continuous with respect to the relative topology on D_x. Then:

(i) *The sets*

$$G = \operatorname{proj}(D) ,$$
$$I = \{x \in G \mid \textit{for some } \ y \in D_x, f(x, y) = \inf f_x\} ,$$

are Borel.

(ii) *For each $\varepsilon > 0$, there is a Borel measurable function φ_ε satisfying, for $x \in G$,*

$$\begin{aligned} f(x, \varphi_\varepsilon(x)) &= \inf f_x , && \textit{if} \quad x \in I , \\ &\leqq \varepsilon + \inf f_x , && \textit{if} \quad x \notin I , \quad \textit{and} \quad \inf f_x \neq -\infty \\ &\leqq -\varepsilon^{-1} , && \textit{if} \quad x \notin I , \quad \textit{and} \quad \inf f_x = -\infty . \end{aligned}$$

PROOF. Set

$$E = \{((x, v), y) \in (X \times R) \times Y \mid (x, y) \in D,\ f(x, y) \leqq v\}$$

where R is the set of real numbers. Then E is Borel, and for each $(x, v) \in X \times R$, the section $\{y \in Y \mid (x, y) \in D, f(x, y) \leqq v\}$ is a countable union of sets compact in Y. By Theorem 1 there is a Borel measurable ρ defined on the Borel set

$$A = \{(x, v) \in X \times R \mid \text{for some } \ y \in Y, ((x, v), y) \in E\},$$

and $((x, v), \rho(x, v)) \in E$ for all $(x, v) \in A$.

The candidate for $\varphi_\varepsilon(x)$, if $x \in I$, is $\rho(x, \inf f_x)$. For, if $x \in I$, the definition of I implies that $\inf f_x$ is a real number and $(x, \inf f_x)$ belongs to A, the domain of ρ. It follows that

$$\psi : x \to \rho(x, \inf f_x), \qquad x \in I,$$

is well defined on I; and from the definition of ρ, $((x, \inf f_x), \psi(x)) \in E$ there. This implies that $f(x, \psi(x)) = \inf f_x$ for all $x \in I$. On I then, φ_ε will be defined to be ψ.

To establish the Borel measurability of ψ first note that G and

$$F = \{x \in G \mid \inf f_x \neq -\infty\}$$

are Borel sets. That G is Borel follows directly from the hypotheses and Theorem 1. The set F is Borel because

$$G - F = \bigcap_{n=1}^{\infty} \{x \in X \mid \text{for some } \ y \in Y, (x, y) \in D \ \text{ and } \ f(x, y) \leqq -n\}$$

and each member of the intersection is the projection of a Borel set of which every x-section, by the lower semi-continuity of f_x, is σ-compact.

Similarly, the set $\{x \in F \mid \inf f_x < a\}$, where a is real, may be expressed as the intersection of F with the countable union:

$$\bigcup_{n=1}^{\infty} \left\{x \in X \mid \text{for some } \ y, (x, y) \in D \ \text{ and } \ f(x, y) \leqq a - \frac{1}{n}\right\}.$$

Each member of the union is Borel, so that the function $x \to \inf f_x$, $x \in F$, is Borel measurable. The easily checked equality $I = \{x \in F \mid (x, \inf f_x) \in A\}$ then implies that I is a Borel set. Finally ψ is Borel measurable, because it is the composition of ρ and the function $x \to (x, \inf f_x)$ which are both Borel measurable.

As mentioned above φ_ε restricted to I will be ψ. To complete the definition of φ_ε, it is slightly more convenient for later arguments, and just as simple for this one, to forget ψ temporarily and begin again. Set

$$E_\varepsilon = \{((x, v), y) \in (X \times R) \times Y \mid (x, y) \in D, f(x, y) \leqq v + \varepsilon\}.$$

Reasoning exactly as before, there exists a Borel measurable ρ_ε defined on the Borel set

$$A_\varepsilon = \{(x, v) \in X \times R \mid \text{for some } \ y \in Y, ((x, v), y) \in E_\varepsilon\}$$

for which $((x, v), \rho_\varepsilon(x, v)) \in E_\varepsilon$ for all $(x, v) \in A_\varepsilon$. Define g on G by

$$\begin{aligned} g(x) &= \inf f_x , & x \in F . \\ &= -(\varepsilon^{-1} + \varepsilon) , & x \in G - F . \end{aligned}$$

Then, if $x \in G$, $(x, g(x)) \in A_\varepsilon$. Again, it follows that

$$\psi_\varepsilon \colon x \to \rho_\varepsilon(x, g(x)) , \qquad x \in G ,$$

is well defined on G; and $((x, g(x)), \psi_\varepsilon(x)) \in E_\varepsilon$ there. Thus, for $x \in G$,

$$\begin{aligned} f(x, \psi_\varepsilon(x)) &\leqq \varepsilon + \inf f_x , & \text{if } \inf f_x \neq -\infty \\ &\leqq -\varepsilon^{-1} , & \text{if } \inf f_x = -\infty . \end{aligned}$$

The measurability of ψ_ε is at hand, so that

$$\begin{aligned} \varphi_\varepsilon(x) &= \psi(x) , & x \in I , \\ &= \psi_\varepsilon(x) , & x \in G - I , \end{aligned}$$

satisfies (ii).

REMARK 1. In Corollary 1 suppose f is permitted to take on the values $+\infty$, $-\infty$, and that for each Borel set B of real numbers, each of the three sets

$$\{z \in D \mid f(z) = +\infty\} , \qquad \{z \in D \mid f(z) = -\infty\} , \qquad \{z \in D \mid f(z) \in B\} ,$$

is Borel. Take lower semi-continuity of $f(x, \cdot)$ to mean that, for each real a, the set $\{y \in D_x \mid f(x, y) \leqq a\}$ is a closed subset of D_x. The usual ordering conventions of the extented real line are assumed to hold.

With these qualifications, the conclusions of the corollary remain true. The proof will not be given here.

REMARK 2. A topological space S is said to be *separable absolute Borel* if there is a complete separable metric space Z such that S is homeomorphic to a member of $\mathscr{B}(Z)$. It is easy to see that the corollary also holds if X, Y are separable absolute Borel sets.

REMARK 3. Suppose that in Corollary 1, D_x is compact for all $x \in \operatorname{proj}(D)$ and Y is k-dimensional Euclidean space. Then $G = I$ and a suitable value of y for which $f(x, y) = \inf f_x$ may be described explicitly in the manner sketched below.

If $k = 1$, simply take $\varphi(x)$ to be the least y in D_x such that $f(x, y) = \inf f_x$. Then the Borel measurability of φ is a consequence of the following result:

> (*) Let E be a Borel set in $X \times R$, where X is a complete separable metric space and R is the real line. If E_x is compact in R, for $x \in X$, then $\operatorname{proj}(E)$ is a Borel set and the function $x \to \inf E_x$, $x \in \operatorname{proj}(E)$ is Borel measurable.

Although (*) is an instance of Corollary 1 (by taking f to be 0 on E and 1 on the complement of E) it is worth noting that it follows via the proof of Theorem

1 from a special case of the theorem of Stschegolkow: if a Borel set of the unit square has the property that each of its vertical sections is closed, that Borel set has a Borel section. This case appears much easier to establish than the theorem.

To indicate why (*) implies the measurability of φ, note first that it implies that proj (D) is Borel. Then the argument given in the fourth paragraph of the proof of Corollary 1 may be repeated to show that $x \to \inf f_x$, $x \in \text{proj } D$, is Borel measurable. This only uses the projection clause of (*). It follows that the set $E = \{(x, y) \in D \mid f(x, y) = \inf f_x\}$ is Borel. Since each x-section of E is compact (*) implies that $\varphi\colon x \to \inf E_x$, $x \in \text{proj } (D)$, is Borel measurable, which finishes the case $k = 1$.

The explicit definition of φ in the case $k = 2$ is accomplished by two iterations of the argument given in the preceding paragraph. First, let

$$D^1 = \{((x, y), z) \mid (x, (y, z)) \in D\}$$

and $f^1((x, y), z) = f(x, (y, z))$ for all $((x, y), z) \in D^1$. Then f^1 satisfies the hypothesis of Corollary 1 and the first paragraph of this remark. Thus, by the preceding paragraph, the set $D^2 = \text{proj } (D^1)$, the function f^2 defined on D^2 by

$$f^2(u) = \inf \{f^1(u, z) \mid z \in D_u^{\,1}\} ,$$

and the function φ^1 defined on D^2 by $\varphi^1(u) =$ least z in $D_u^{\,1}$ such that

$$f^1(u, z) = f^2(u) ,$$

are all Borel measurable. In addition, for $x \in \text{proj } (D^2)$ $D_x^{\,2}$ is compact and $f^2(x, \cdot)$ is lower semi-continuous on $D_x^{\,2}$, so that f^2 satisfies all the hypotheses of Corollary 1 and the first paragraph of this remark. Again, from the case $k - 1$, proj (D^2) is Borel and the function φ^2, defined on proj (D^2) by $\varphi^2(x) =$ least y in $D_x^{\,2}$ such that $f(x, y) = \inf f_x^{\,2}$ is Borel measurable. Then the Borel measurable $\varphi : x \to (\varphi^2(x), \varphi^1(x, \varphi^2(x)))$ will serve to pick a minimizing (y, z) for each $x \in \text{proj } (D^2)(= \text{proj } (D))$; that is, $f(x, \varphi(x)) = \inf \{f(x, v) \mid v \in D_x\}$. The inclusion of domains required for the definition of the composition φ is easily checked, as is the fact that $\varphi(x)$ is a minimizing (y, z) for each x.

Although omitted here, an inductive argument, similar to the above step from $k = 1$ to $k = 2$, will establish a recipe for an explicitly defined φ for each positive integer k.

3. Absolute measurability. Before proceeding to the result of this section certain properties of analytic sets and absolutely measurable sets must be introduced. A non-empty subset of a complete separable metric space U is *analytic* if it is the range of a continuous function defined on the subspace of irrational numbers in $[0, 1]$. The empty set is taken to be analytic. The sigma-field generated by the collection of analytic sets in U will be denoted $\Sigma(U)$. A subset of a metric space M is *absolutely measurable in M* if it belongs to the domain of the completion of every finite measure on $\mathscr{B}(M)$.

The following two results will be required. Let U, V be complete, separable metric spaces.

I. If B is a Borel subset of $U \times V$, proj (B) is analytic (Kuratowski (1958) page 353).

II. If A is an analytic set in U, A is absolutely measurable in U (Kuratowski (1958) page 391).

Throughout the remainder of this section, the following (somewhat anomalous) definitions will be used. Let h be a function whose domain is a subset of a metric space M and which takes its values in some metric space N. Then h will be said to be *absolutely measurable in M* if $h^{-1}(B)$ is an absolutely measurable set in M for all $B \in \mathscr{B}(N)$. Similarly, h will be said to be *measurable with respect to* $\Sigma(M)$ if $h^{-1}(B) \in \Sigma(M)$ for all $B \in \mathscr{B}(N)$. In the case that the domain of h coincides with M, these definitions are the usual ones.

LEMMA. *Let M, N be metric spaces and h be a function with domain $h \subseteqq M$ and range $h \subseteqq N$. Suppose that h is absolutely measurable in M. Then if L is an absolutely measurable set in N, $h^{-1}(L)$ is an absolutely measurable set in M.*

PROOF. The proof is easy and is omitted.

THEOREM 2. *Let X, Y be complete separable metric spaces and f be a real-valued Borel measurable function defined on a Borel subset D of $X \times Y$. Then*

(i) *The sets*

$$G = \operatorname{proj}(D)\,,$$
$$I = \{x \in G \mid \textit{for some } \ y \in D_x \ \ f(x, y) = \inf f_x\}\,,$$

are absolutely measurable.

(ii) *For each $\varepsilon > 0$, there is an absolutely measurable function φ_ε satisfying, for $x \in G$,*

$$\begin{aligned} f(x, \varphi_\varepsilon(x)) &= \inf f_x\,, && \textit{if} \quad x \in I\,, \\ &\leqq \varepsilon + \inf f_x && \textit{if} \quad x \notin I\,, \quad \textit{and} \quad \inf f_x \neq -\infty\,, \\ &\leqq -\varepsilon^{-1}\,, && \textit{if} \quad x \notin I\,, \quad \textit{and} \quad \inf f_x = -\infty\,. \end{aligned}$$

PROOF. Excluding considerations of measurability, the proof is the same as the proof of Corollary 1. All that will be done here is to show the absolute measurability of the various sets and functions defined in the proof of Corollary 1.

The set E is Borel and the set A is analytic, being the projection of E. Then the following selection theorem applies to show that ρ may be taken to be measurable with respect to $\Sigma(X \times R)$: If E is any Borel (or even analytic) set in $U \times V$, where U, V are complete separable metric, there is a set S satisfying (ii), (iii), (iv) of the definition of Borel selection given in Section 2 of this paper, such that ρ_s is measurable with respect to $\Sigma(U)$. In the case that U is the real line, this selection theorem follows from a lemma of Von Neumann (Von

Neumann (1949) Lemma 5). The lemma is easily extended to an arbitrary (uncountable) complete separable U by using the Borel isomorphism of U and the real line. (The countable case is trivial.) The selection theorem is also a special case of a theorem of Sion (Sion (1960) Corollary 4.4).

To establish the absolute measurability of ψ first note that both G and F belong to $\Sigma(X)$. The set G is analytic, being the projection of the Borel set D, and so belongs to $\Sigma(X)$. The set F belongs to $\Sigma(X)$ because

$$G - F = \bigcap_{n=1}^{\infty} \{x \in X \mid \text{for some } \; y, (x, y) \in D \;\text{ and }\; f(x, y) \leqq -n\}$$

and each member of the intersection is the projection of a Borel set. Similarly, the equality

$$\{x \in F \mid \inf f_x < a\} = \{x \in X \mid \text{for some } \; y, (x, y) \in D \;\text{ and }\; f(x, y) < a\} \cap F$$

where a is real, shows that the set on the left belongs to $\Sigma(X)$. Thus the function $x \to \inf f_x$, $x \in F$, is measurable with respect to $\Sigma(X)$, and by a standard argument, so is the function

$$T: x \to (x, \inf f_x)\,, \qquad x \in F\,.$$

Observe that, for $x \in I$, $\psi(x) = \rho(T(x))$. Then the equality

$$I = \{x \in F \mid (x, \inf f_x) \in A\}$$

implies that $\psi^{-1}(B) = T^{-1}(A \cap \rho^{-1}(B))$ for every subset B of Y. If B is a Borel set in Y, $\rho^{-1}(B) \in \Sigma(X \times R)$ and $A \cap \rho^{-1}(B) \in \Sigma(X \times R)$. Thus to show ψ is absolutely measurable in X, it is sufficient to show that for $L \in \Sigma(X \times R)$, $T^{-1}(L)$ is absolutely measurable in x. The function T has already been shown to be measurable with respect to $\Sigma(X)$. The statement II implies that all members of $\Sigma(X)$ are absolutely measurable in X, so that T is absolutely measurable in X. Again by II, all members of $\Sigma(X \times R)$ are absolutely measurable in $X \times R$. The lemma then applies to show that, for $L \in \Sigma(X \times R)$, $T^{-1}(L)$ is absolutely measurable in X. Thus ψ is absolutely measurable in X. This implies that I, the domain of ψ, is absolutely measurable. Also G is absolutely measurable by II.

The arguments just given may be applied to show that the function ψ_ε of the last paragraph of the proof of Corollary 1 is also absolutely measurable in X. This completes the proof of Theorem 2.

REMARK 1. Suppose f is permitted to take on the values $+\infty$, $-\infty$ and is Borel measurable in the sense of the first remark in Section 2. Then the conclusions of Theorem 2 remain true. The proof will be omitted.

REMARK 2. Theorem 2 also holds if X, Y are separable, absolute Borel sets and D is only assumed to belong to $\mathscr{B}(X \times Y)$. The proof is straightforward and is omitted.

4. Measurability of Bayes procedures. In the statement of the next theorem, certain assumptions will be required concerning the elements of the statistical problem at hand. These are presented as Assumptions 1–3. Only the fixed sample size problem is considered here, although the results have some relevance to the sequential case.

ASSUMPTION 1. The sample space $\mathscr{X}$ and the space of possible decisions $\mathscr{A}$ are non-empty Borel sets (in complete separable metric spaces).

ASSUMPTION 2. Let $(\Theta, \mathscr{C}, G)$ be a probability space, where G is considered to be the prior distribution on the parameter space Θ. Suppose that, for each $\theta \in \Theta$, P_θ is a probability distribution defined on $\mathscr{B}(\mathscr{X})$, such that, for each $B \in \mathscr{B}(\mathscr{X})$, the function $\theta \to P_\theta(B)$, $\theta \in \Theta$, is measurable with respect to $\mathscr{C}$. Suppose further that there is a conditional distribution $P(\cdot \mid x)$, $x \in \mathscr{X}$, satisfying:

(i) For each $x \in \mathscr{X}$, $P(\cdot \mid x)$ is a probability on $(\Theta, \mathscr{C})$.
(ii) For $C \in \mathscr{C}$, $P(C \mid \cdot)$ is a Borel measurable function on $\mathscr{X}$.
(iii) If P is defined on the Borel subsets of $\mathscr{X}$ by the equation

$$P(\cdot) = \int P_\theta(\cdot) G(d\theta) ,$$

then

$$\int_B P(C \mid x) P(dx) = \int_C P_\theta(B) G(d\theta)$$

for all $C \in \mathscr{C}$ and $B \in \mathscr{B}(\mathscr{X})$.

ASSUMPTION 3. The loss function L, defined on $\Theta \times \mathscr{A}$ and taking nonnegative reals as values, is measurable with respect to the product of the sigma-field $\mathscr{C}$ and the Borel sigma-field in $\mathscr{A}$.

REMARK. In Assumption 2, the existence of the family $P(\cdot \mid x)$ is assured if Θ is a Borel set and $\mathscr{C}$ is the class of Borel subsets of Θ. This follows from standard conditional probability theorems (such as Doob (1953) Theorem I.9.5).

In the statement of Theorem 3 below define r, the conditional risk function on $\mathscr{X} \times \mathscr{A}$, by

$$r(x, a) = \int_\Theta L(\theta, a) P(d\theta \mid x) ,$$

and set

$$I = \{x \in \mathscr{X} \mid \text{for some } b \in \mathscr{A},\ r(x, b) = \inf_{a \in \mathscr{A}} r(x, a)\} .$$

THEOREM 3. *Suppose Assumptions 1–3 are satisfied.*

(i) *For each $\varepsilon > 0$, there is an absolutely measurable decision procedure d_ε, satisfying, for $x \in \mathscr{X}$,*

$$\begin{aligned} r(x, d_\varepsilon(x)) &= \inf_{a \in \mathscr{A}} r(x, a) , && \text{if } x \in I , \\ r(x, d_\varepsilon(x)) &\leqq \varepsilon + \inf_{a \in \mathscr{A}} r(x, a) , && \text{if } x \in \mathscr{X} - I . \end{aligned}$$

(ii) *If $\mathscr{A}$ is a countable union of compact sets and $L(\theta, \cdot)$ is lower semi-continuous on $\mathscr{A}$ for each $\theta \in \Theta$, the d_ε of* (i) *may be taken to be Borel measurable.*

PROOF. The sets $\{d \in \text{domain } r \mid r(d) = +\infty\}$, $\{d \in \text{domain } r \mid r(d) \in B\}$ where B is a linear Borel set, are Borel. This follows from the three assumptions by the usual measure-theoretic arguments. Thus the first remark of Section 3 implies (i).

In (ii), Fatou's lemma shows that for each $x \in \mathscr{X}$, $r(x, \cdot)$ is lower semi-continuous on $\mathscr{A}$. Thus the first remark of Section 2 implies (ii).

REMARKS. If $\mathscr{X}$, $\mathscr{A}$ are only assumed to be separable, absolute Borel sets, Theorem 3 is also true according to Remarks 2.2, 3.2.

5. A counterexample. If the sections E_u in Theorem 1 are merely closed sets, rather than compact sets, the conclusion of the theorem may not hold. As is shown in the example to follow, this is an obvious consequence of the (not obvious) fact that an analytic set exists which is not Borel (Kuratowski (1958) page 368).

Let J be the subspace of the irrationals in $[0, 1]$ metrized so that it becomes a complete separable metric space. Set $U = [0, 1]$, $V = J$. Let A be an analytic set in $[0, 1]$ which is not Borel and let f be a continuous function from J onto A. Let $E = \{(u, v) \in U \times V \mid f(v) = u\}$. Since E is the graph in $(V \times U)$ of the Borel measurable function f, E is a Borel set (see e.g. Halmos (1950) page 143). Also $E_u = \{v \in V \mid f(v) = u\}$ is closed, $u \in U$. However, $\operatorname{proj}(E) = A$ is not Borel, so that the second conclusion of Theorem 1 fails. Nor does E admit a Borel selection. For if it did, $\operatorname{proj}(E)$ would be Borel as shown in the proof of Theorem 1.

This example may be modified to show that part (ii) of Theorem 3 will fail if $\mathscr{A}$ is only assumed to be closed. Set $\mathscr{X} = \Theta = [0, 1]$, $\mathscr{A} = J$ $G =$ Lebesgue measure on $[0, 1]$, $P_\theta =$ point mass at θ, and

$$\begin{aligned} L(\theta, a) &= 0 , \qquad \text{if} \quad (\theta, a) \in E , \\ &= 1 , \qquad \text{if} \quad (\theta, a) \notin E . \end{aligned}$$

Note that $L(\theta, \cdot)$ is lower semi-continuous.

A version of the posterior $P(\cdot \mid x)$ is point mass at x, and for it $r = L$. Suppose a Borel measurable d exists on $\mathscr{X}$ such that $r(x, d(x)) = \inf_a r(x, a)$. Then, the set on the left, and so the set on the right of the equality

$$\{x \in \mathscr{X} \mid r(x, d(x)) = 0\} = \{x \in \mathscr{X} \mid \inf_a r(x, a) = 0\}$$

would be Borel. But the set on the right is $\operatorname{proj}(E)$, which is not Borel. A similar argument shows that, for $\varepsilon < 1$, no Borel measurable d exists on $\mathscr{X}$ for which $r(x, d(x)) \leqq \inf_a r(x, a) + \varepsilon$, $x \in \mathscr{X}$.

A natural question to ask of this example is whether or not it would be possible to choose another version of the posterior for which the r so obtained admits of a Borel measurable d. The answer is yes, for the general reasons outlined in the next paragraph.

Suppose Assumptions 1, 2, 3 of the preceding section are in force. Let $\varepsilon > 0$. Applying (i) of Theorem 3, there is an absolutely measurable d_ε satisfying the conditions stated there. Then there is a Borel set N in $\mathscr{X}$ with $P(N) = 0$ and a Borel measurable function $h: \mathscr{X} \to \mathscr{A}$ such that $d_\varepsilon(x) = h(x)$ for all $x \in \mathscr{X} - N$. Fix a point $u \in N$ and set $\hat{P}(\cdot \mid x) = P(\cdot \mid x)$ for $x \in \mathscr{X}$-N and

$\hat{P}(\cdot \mid x) = P(\cdot \mid u)$ for $x \in N$. Then $\hat{P}(\cdot \mid x)$, $x \in \mathscr{X}$, satisfies (i)–(iii) of Assumption 2. Let $\hat{r}$ be the conditional risk function associated with $\hat{P}(\cdot \mid x)$, $x \in \mathscr{X}$, and $\hat{I}$ be the set of $x \in \mathscr{X}$ for which a $b \in \mathscr{A}$ exists such that $\hat{r}(x, b) = \inf_{a \in \mathscr{A}} \hat{r}(x, u)$. Then, as is easily verified, the decision function defined by

$$\begin{aligned} \hat{d}_\varepsilon(x) &= h(x)\,, \qquad x \in \mathscr{X} - N\,, \\ &= d_\varepsilon(u)\,, \qquad x \in N\,, \end{aligned}$$

is Borel measurable, and satisfies the equality and inequality of Theorem 3(i) (with $\hat{r}$, $\hat{I}$ substituted for r, I resp.).

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